

MONOPOLE CLASSES AND PERELMAN'S INVARIANT OF FOUR-MANIFOLDS

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ABSTRACT. We calculate Perelman's invariant for compact complex surfaces and a few other smooth four-manifolds. We also prove some results concerning the dependence of Perelman's invariant on the smooth structure.

1. INTRODUCTION

In his celebrated work on the Ricci flow [30, 31], G. Perelman introduced an interesting invariant of closed manifolds of arbitrary dimension. By definition, Perelman's invariant is closely related to the Yamabe invariant or sigma constant. For three-manifolds that do not admit a metric of positive scalar curvature, Perelman's work [30, 31] shows that his invariant is equivalent to M. Gromov's minimal volume [13], which is a priori a very different kind of invariant.

In this paper we calculate Perelman's invariant for compact complex surfaces and show that it essentially coincides with the Yamabe invariant:

Theorem 1. *Let Z be a minimal complex surface with $b_1(Z)$ even and with $b_2^+(Z) > 1$. Then Perelman's invariant for the manifold $X = Z \# k\mathbb{C}P^2 \# l(S^1 \times S^3)$ is given by*

$$(1) \quad \bar{\lambda}_X = -4\pi\sqrt{2c_1^2(Z)}.$$

The supremum defining $\bar{\lambda}_X$ is realized by a metric if and only if $k = l = 0$ and Z admits a Kähler–Einstein metric of non-positive scalar curvature.

We also give calculations and estimates for some other classes of four-manifolds, see in particular Theorems 4, 6, and 7. Our results are applications, and, in some cases, generalizations of the recent results of F. Fang and Y. Zhang [7], who discovered a relationship between Perelman's invariant and solutions of the Seiberg–Witten monopole equations on four-manifolds. In particular, Fang and Zhang [7] already noted that Perelman's invariant is not a homeomorphism invariant. Our Theorems 6, 7 and 8 contain quantitative results elaborating on this observation.

As a consequence of Seiberg–Witten theory it is now well known that the Yamabe invariant is in fact sensitive to the smooth structure of a four-manifold. As was pointed out in [21], this is also true for Gromov's minimal volume, although this is harder to prove than for the Yamabe invariant. In Theorem 7 below we prove that the vanishing or non-vanishing of the minimal volume does depend on the smooth structure, and then discuss the relationship between this result and Perelman's invariant.

2. PERELMAN'S INVARIANT

We recall the basic definition from [30], compare also [17, 7].

Let M be a closed oriented manifold of dimension $n \geq 3$. For a Riemannian metric g on M and a function $f \in C^\infty(M)$, Perelman defines

$$\mathcal{F}(g, f) = \int_M (s_g + |\nabla f|^2) e^{-f} d\text{vol}_g ,$$

where s_g is the scalar curvature of g . Then Perelman's invariant of the Riemannian metric g is

$$\lambda_M(g) = \inf_{f \in C^\infty(M)} \{ \mathcal{F}(g, f) \mid \int_M e^{-f} d\text{vol}_g = 1 \} .$$

This infimum is actually a minimum, because it coincides with the smallest eigenvalue of the operator $d^*d + s_g$. It follows that $\lambda_M(g)$ depends continuously on g . If the scalar curvature s_g is constant, then $\lambda_M(g) = s_g$.

The quantity

$$\bar{\lambda}_M(g) = \lambda_M(g) \cdot \text{Vol}(M, g)^{2/n}$$

is scale invariant, and can be used to define a diffeomorphism invariant of M by setting

$$(2) \quad \bar{\lambda}_M = \sup_g \bar{\lambda}_M(g) .$$

We shall call $\bar{\lambda}_M$ Perelman's invariant of M .

If the supremum in (2) is achieved, then the corresponding metric has to be Einstein, cf. [30, 17]. We will see that very often this is not possible, so that the supremum is not a maximum.

Recall that for a conformal class C on M , the Yamabe invariant of C is defined by

$$\mu_M(C) = \inf_{g \in C} \frac{\int_M s_g d\text{vol}_g}{\text{Vol}(M, g)^{(n-2)/n}} .$$

By the solution to the Yamabe problem due to H. Yamabe, N. Trudinger, T. Aubin and R. Schoen, this infimum is always realized by some metric g_0 of constant scalar curvature s_0 . Such a metric is called a Yamabe metric. For a Yamabe metric $g_0 \in C$ we have

$$\mu_M(C) = \frac{\int_M s_0 d\text{vol}_{g_0}}{\text{Vol}(M, g_0)^{(n-2)/n}} = s_0 \cdot \text{Vol}(M, g_0)^{2/n} = \bar{\lambda}_M(g_0) .$$

The Yamabe invariant of M is defined as

$$\mu_M = \sup_C \mu_M(C) .$$

By the above calculation this can be written as

$$\mu_M = \sup \{ \bar{\lambda}_M(g_0) \mid g_0 \text{ a Yamabe metric} \} .$$

Comparing this with the definition (2) of Perelman's invariant, we obtain the fundamental inequality

$$(3) \quad \mu_M \leq \bar{\lambda}_M$$

between the Yamabe and Perelman invariants of M .

3. MONOPOLE CLASSES

Consider now a closed smooth oriented four-manifold X with a Spin^c -structure \mathfrak{s} . For every choice of Riemannian metric g , the Seiberg–Witten monopole equations for (X, \mathfrak{s}) with respect to g are a system of coupled equations for a pair (A, Φ) , where A is a Spin^c -connection in the spin bundle for \mathfrak{s} and Φ is a section of the positive spin bundle V_+ . The equations are:

$$(4) \quad D_A^+ \Phi = 0 ,$$

$$(5) \quad F_{\hat{A}}^+ = \sigma(\Phi, \Phi) ,$$

with D_A^+ the half-Dirac operator defined on spinors of positive chirality, and \hat{A} the connection induced by A on the determinant of the spin bundle. The right-hand side of the curvature equation (5) is the 2-form which, under the Clifford module structure determined by \mathfrak{s} , corresponds to the trace-free part of $\Phi \otimes \Phi^*$.

Solutions (A, Φ) with $\Phi \equiv 0$ are called *reducible*. If there is a reducible solution, then $c = c_1(\mathfrak{s})$ is represented by an anti-self-dual harmonic form because of (5). This implies $c^2 \leq 0$, with equality if and only if c is a torsion class.

The following definition is due to P. Kronheimer [23], see also [20].

Definition 1. A class $c \in H^2(X, \mathbb{Z})$ is called a *monopole class*, if there is a Spin^c -structure \mathfrak{s} on X with $c = c_1(\mathfrak{s})$ for which the monopole equations (4) and (5) admit a solution (A, Φ) for every choice of metric g .

Of course, on manifolds for which the Seiberg–Witten invariants are well-defined, every basic class is a monopole class. The rationale for considering the concept of a monopole class is that the existence of solutions to the monopole equations has immediate consequences, even when the corresponding invariants vanish.

The following result from [7] establishes an important relation between the Seiberg–Witten equations and Perelman’s invariant:

Proposition 1 (Fang–Zhang [7]). *If the monopole equations (4) and (5) for (X, \mathfrak{s}) with respect to g admit an irreducible solution (A, Φ) , then*

$$(6) \quad \bar{\lambda}_X(g) \leq -4\pi\sqrt{2(c_1^+)^2} ,$$

where c_1^+ denotes the projection of $c_1(\mathfrak{s})$ to the g -self-dual subspace of $H^2(X; \mathbb{R})$.

If $c_1^+ \neq 0$, then equality in (6) can only occur if g is a Kähler metric of constant negative scalar curvature.

The proof is an adaptation of the usual scalar curvature estimate for solutions of the Seiberg–Witten equations obtained by combining the Bochner–Weitzenböck formula with the equations, cf. [35].

The following is a slight generalization of Theorem 1.1 in [7]:

Theorem 2. *Let X be a smooth closed oriented 4-manifold with a monopole class c that is not a torsion class and satisfies $c^2 \geq 0$. Then*

$$(7) \quad \bar{\lambda}_X \leq -4\pi\sqrt{2c^2} .$$

Proof. Because c is assumed to be a monopole class, there is a Spin^c -structure \mathfrak{s} with $c_1(\mathfrak{s}) = c$ such that the monopole equations for \mathfrak{s} have a solution (A, Φ) for every choice of metric g . If the solution is irreducible, i. e. $\Phi \neq 0$, then (6) implies

$$\bar{\lambda}_X(g) \leq -4\pi\sqrt{2(c^+)^2} \leq -4\pi\sqrt{2c^2}.$$

If $c^2 > 0$, then it is clear that all solutions are irreducible. If $c^2 = 0$, then we use the assumption that c is not a torsion class, to argue that solutions must be irreducible for generic g . In fact, this follows from:

Lemma 1 (Donaldson). *Let $c \in H^2(X, \mathbb{Z})$ be a non-torsion class. If $b_2^+(X) > 0$, then for a generic metric g , there is no anti-self-dual harmonic form representing the image of c in $H^2(X, \mathbb{R})$.*

A proof of the lemma can be found in [6]. In our case the intersection form must be indefinite because there is a non-torsion class of square zero, and thus $b_2^+(X) > 0$ holds. Therefore we obtain the desired estimate for generic metrics g . As Perelman's invariant $\bar{\lambda}_X(g)$ depends continuously on g , the estimate holds for all g . \square

Next we adapt the proof of Theorem 4.2 in [20] to show that in the presence of a monopole class, Perelman's invariant can be used to bound the number of smooth exceptional spheres in a four-manifold.

Theorem 3. *Let X be a smooth four-manifold with a monopole class c . The maximal number k of copies of $\overline{\mathbb{C}P^2}$ that can be split off smoothly is bounded by*

$$(8) \quad k \leq \frac{1}{32\pi^2} \bar{\lambda}_X^2 - c^2.$$

Proof. Suppose that $X \cong Y \# k\overline{\mathbb{C}P^2}$, with $k > 0$, and write $c = c_Y + \sum_{i=1}^k a_i e_i$, with respect to the obvious direct sum decomposition of $H^2(X, \mathbb{Z})$. Here e_i are the generators for the cohomology of the $\overline{\mathbb{C}P^2}$ summands. Note that the a_i are odd integers because c must be characteristic. In particular c can not be a torsion class.

Now the reflections in the e_i are realised by self-diffeomorphisms of X , and the images of our monopole class under these diffeomorphisms are again monopole classes. Thus, moving c by a diffeomorphism, we can arbitrarily change e_i to its negative.

Given a metric g on X , we choose the signs in such a way that $a_i e_i^+ \cdot c_Y^+ \geq 0$. Then we find

$$(c^+)^2 = \left(c_Y^+ + \sum_{i=1}^k a_i e_i^+ \right)^2 \geq (c_Y^+)^2 \geq c_Y^2 = c^2 + \sum_{i=1}^k a_i^2.$$

If g is generic, then there are irreducible solutions to the monopole equations, and applying (6) to c and g gives

$$(c^+)^2 \leq \frac{1}{32\pi^2} (\bar{\lambda}_X(g))^2.$$

Combining the two inequalities and noting that $a_i^2 \geq 1$ because all the a_i are odd integers shows

$$k \leq \frac{1}{32\pi^2} (\bar{\lambda}_X(g))^2 - c^2$$

for generic g . By continuity of Perelman's invariant this holds for all g , giving (8). \square

It may not be obvious why this Theorem is interesting, but this should become clear by looking at the following special case:

Theorem 4. *Let Z be a minimal symplectic four-manifold with $b_2^+(Z) > 1$, and $X = Z \# k\overline{\mathbb{C}P^2} \# l(S^1 \times S^3)$. Then*

$$(9) \quad \bar{\lambda}_X \leq -4\pi\sqrt{2c_1^2(Z)}.$$

The case $l = 0$ was previously proved by Fang and Zhang in Theorem 1.4 of [7].

Proof. Let $Y = Z \# k\overline{\mathbb{C}P^2}$, which we can think of as a symplectic blowup. By the result of C. Taubes [33], the first Chern class $c_1(Y)$ of a symplectic structure is a Seiberg–Witten basic class with numerical Seiberg–Witten invariant ± 1 . In particular, it is a monopole class on Y .

Now consider $X = Y \# l(S^1 \times S^3)$. Although its numerical Seiberg–Witten invariants must vanish, cf. [22, 18], we claim that each of the basic classes with numerical Seiberg–Witten invariant $= \pm 1$ on Y gives rise to a monopole class on Z . There are two ways to see this. One can extract our claim from the connected sum formula [3] for the stable cohomotopy refinement of Seiberg–Witten invariants introduced by S. Bauer and M. Furuta [4], cf. [12]. Alternatively, one uses the invariant defined by the homology class of the moduli space of solutions to the monopole equations, as in [18]. This means that the first homology of the manifold is used, and here this is enough to obtain a non-vanishing invariant. Using this invariant, our claim follows from Proposition 2.2 of P. Ozsváth and Z. Szabó [26]. See also K. Froyshov [11].

We apply (8) to X with the monopole class $c_1(Y) \in H^2(Y; \mathbb{Z}) = H^2(X; \mathbb{Z})$ to obtain

$$k \leq \frac{1}{32\pi^2} \bar{\lambda}_X^2 - c_1^2(Y) = \frac{1}{32\pi^2} \bar{\lambda}_X^2 - c_1^2(Z) + k.$$

As $\bar{\lambda}_X$ is non-positive in this case, the claim follows. \square

In this argument the expected dimension of the moduli space of solutions to the Seiberg–Witten equations is $= 0$ on Y , but is $= l$ on X . As explained in [20], results like Theorem 3 are stronger the larger the expected dimension of the moduli space is.

In the case of Kählerian complex surfaces we can combine this upper bound for Perelman’s invariant with the lower bound given by the Yamabe invariant to obtain a complete calculation.

Proof of Theorem 1. Every compact complex surface Z with even first Betti number admits a Kähler structure, and is therefore symplectic. For simplicity we are assuming $b_2^+(Z) > 1$, so that surfaces of negative Kodaira dimension do not occur. For Z with a Kähler structure of non-negative Kodaira dimension, holomorphic and symplectic minimality coincide, cf. [14]. Thus Z is symplectically minimal and we can apply Theorem 4 to obtain

$$(10) \quad \bar{\lambda}_X \leq -4\pi\sqrt{2c_1^2(Z)}.$$

For the reverse inequality consider first the case $k = l = 0$, i. e. $X = Z$. Then we have $\bar{\lambda}_Z \geq \mu_Z$ by (3), and, if Z is of general type, then the Yamabe invariant μ_Z equals $-4\pi\sqrt{2c_1^2(Z)}$ by the result of C. LeBrun [24]. This is easy to see when Z admits a Kähler–Einstein metric of negative scalar curvature, cf. [24, 7]. In the case when such a metric does not exist, one has to consider sequences of metrics which suitably approximate an orbifold Kähler–Einstein metric on the canonical model of Z , see [24]. If Z is not of general type, then $c_1^2(Z) = 0$, and Z is either a $K3$ surface, an elliptic surface, or an Abelian surface, cf. [2]. In all these cases Z does not admit a metric of positive scalar curvature, but it does collapse with bounded scalar curvature, in fact even with bounded Ricci curvature, see LeBrun [25]. Collapsing with bounded scalar curvature can also be seen from the result of G. Paternain and J. Petean [28] that Z has an \mathcal{F} -structure. Because Z collapses with

bounded scalar curvature, its Yamabe and Perelman invariants vanish. This completes the proof of (1) in the case $k = l = 0$.

Now we allow k and l to be positive. By Proposition 4.1 of Fang and Zhang [7], Perelman's invariant does not decrease under connected sum with $\overline{\mathbb{C}P^2}$ and with $S^1 \times S^3$. As the upper bound (10) is achieved on Z and is unchanged by the connected sum, we conclude that it is an equality for all positive k and l as well.

Finally we discuss the question whether the supremum in the definition of $\overline{\lambda}_X$ is a maximum. If this is the case, then the corresponding metric on X is an Einstein metric. If Z is of general type, then $c_1^2(Z) > 0$, and we can use the discussion of the equality case in Proposition 1 to conclude that the critical metric on X is Kähler as well, and the scalar curvature is negative. This implies $k = l = 0$. If Z is not of general type then $c_1^2(Z) = 0$. Now the Hitchin–Thorpe inequality [16] for the Einstein metric implies $k = l = 0$. We are then in the case of equality of the Hitchin–Thorpe inequality, and $X = Z$ is Ricci-flat Kähler. \square

Remark 1. Instead of using the behaviour of Perelman's invariant under connected sum, one can alternatively argue with the corresponding result for the Yamabe invariant (and for \mathcal{F} -structures).

Remark 2. Theorems 4 and 1 have extensions to the case of manifolds with $b_2^+ = 1$. For the latter one also has to consider rational and ruled symplectic manifolds, which do admit metrics of positive scalar curvature and therefore have positive Perelman invariant.

4. EXAMPLES AND APPLICATIONS

In this section we give some examples illustrating the estimates and calculations of Perelman's invariant, with special emphasis on its dependence on the smooth structure.

First, we have the following:

Theorem 5. *The number of distinct values that Perelman's invariant can take on the smooth structures in a fixed homeomorphism type of simply connected four-manifolds is unbounded.*

Proof. By the standard geography results for minimal surfaces of general type going back to U. Persson [32], compare also [2], we can do the following: for every positive integer n we find positive integers x and y with the properties that all pairs $(x - i, y + i)$ with i running from 1 to n are realized as $(c_2(Z_i), c_1^2(Z_i))$ for some simply connected minimal complex surface Z_i of general type. Let X_i be the i -fold blowup of Z_i . Then all the X_i for i from 1 to n are simply connected and non-spin and have the same Chern numbers (x, y) . Therefore, by M. Freedman's result [10] they are homeomorphic to each other. However, by the above Theorem 1, the X_i have pairwise different Perelman invariants. \square

While this construction does produce arbitrarily large numbers of distinct Perelman invariants among homeomorphic four-manifolds, it can never produce infinitely many. Of course it is now easy to construct manifolds with infinitely many distinct smooth structures admitting symplectic forms. However, Theorem 4 does not seem to be strong enough to show that their Perelman invariants take on infinitely many values. Therefore, we leave open the following:

Conjecture 1. *On a suitable homeomorphism type, the Perelman invariant takes on infinitely many distinct values.*

There are also spin manifolds with smooth structures with several distinct Perelman invariants:

Theorem 6. *The manifold $X = 3K3\#4(S^2 \times S^2)$ has:*

- a smooth structure X_0 with $\bar{\lambda}_{X_0} = 0$,
- a smooth structures X_1 with $\bar{\lambda}_{X_1} = -16\pi$,
- infinitely many smooth structures X_i with $\bar{\lambda}_{X_i} \leq -16\sqrt{2}\pi < -16\pi$.

The supremum defining the Perelman invariant is attained for X_1 , but not for X_0 and the X_i .

Proof. The smooth structure X_0 is the standard one given by the connected sum. By the Lichnerowicz argument it has no metric of positive scalar curvature. As it does collapse with bounded scalar curvature, we conclude $\bar{\lambda}_{X_0} = 0$. If this supremum were attained, then the corresponding metric would have to be Ricci-flat. As the signature is non-zero, we would have a parallel harmonic spinor, showing that the manifold is Kähler, which is clearly not possible¹.

The smooth structure X_1 underlies the complex algebraic surface obtained as the double cover of the projective plane branched in a smooth holomorphic curve of degree 10. This is homeomorphic to X by [10]. By Theorem 1, $\bar{\lambda}_{X_1} = -16\pi$. As the canonical bundle of X_1 is ample, X_1 has a Kähler–Einstein metric of negative scalar curvature by the results of T. Aubin [1] and S.-T. Yau [36] on the Calabi conjecture. This metric achieves the supremum for the Perelman invariant.

The smooth structures X_i are constructed as follows, cf. [20]. Let M be a symplectic spin manifold with $c_1^2(M) = 16$ and $\chi(M) = 4$, where $\chi = \frac{1}{4}(e + \sigma)$ denotes the holomorphic Euler characteristic. Such manifolds exist by the results of D. Park and Z. Szabó [27]. By Freedman’s classification [10], such an M is homeomorphic to $K3 \# 4(S^2 \times S^2)$. Take $N = K3$, and O the symplectic spin manifold obtained from $K3$ by performing a logarithmic transformation of odd multiplicity i . By the connected sum formula for the stable cohomotopy refinement of the Seiberg–Witten invariants due to Bauer and Furuta [4, 3], the connected sum $M \# N \# O$ has monopole classes c which are the sums of the basic classes on the different summands. Note that $c^2 = 16$. Therefore $\bar{\lambda}_{X_i} \leq -16\sqrt{2}\pi$ by Theorem 4. It was shown in [20] that as we increase i , the multiplicity of the logarithmic transformation, we do indeed get infinitely many distinct smooth structures. It was also shown in [20] that the X_i do not admit any Einstein metrics. Therefore, the supremum for the Perelman invariant can not be achieved for them. \square

Remark 3. The manifold X has another infinite sequence of smooth structures, which are distinct from the ones discussed above. R. Fintushel and R. Stern [8] have shown that one can perform cusp surgery on a torus in S to construct infinitely many distinct smooth structures which are irreducible and non-complex, and are therefore distinct from the smooth structures we consider. These smooth structures have negative Perelman invariants, and it is unknown whether they admit Einstein metrics.

One can give many similar examples on larger manifolds. We can even obtain interesting results for parallelizable manifolds:

Theorem 7. *For every $k \geq 0$ the manifold $X_k = k(S^2 \times S^2) \# (1+k)(S^1 \times S^3)$ with its standard smooth structure has zero minimal volume and Perelman invariant $\bar{\lambda}_{X_k} = +\infty$.*

If k is odd and large enough, then there are infinitely many pairwise non-diffeomorphic smooth manifolds Y_k homeomorphic to X_k , all of which have strictly positive minimal volume and strictly negative Perelman invariant. Moreover, the supremum in the definition of the Perelman invariant is not achieved. All the Y_k have the property that $Y_k \# (S^2 \times S^2)$ is diffeomorphic to $X_k \# (S^2 \times S^2)$.

¹See the appendix to [20] for details of this argument.

Proof. Note that $X_0 = S^1 \times S^3$ has obvious free circle actions, and therefore collapses with bounded sectional curvature. To see that all X_k have vanishing minimal volume it suffices to construct fixed-point-free circle actions on them, cf. M. Gromov [13].

The product $S^2 \times S^2$ has a diagonal effective circle action which on each factor is rotation around the north-south axis. It has four fixed points, and the linearization of the action induces one orientation at two of the fixed points, and the other orientation at the remaining two. The induced action on the boundary of an S^1 -invariant small ball around each of the fixed points is the Hopf action on S^3 . By taking equivariant connected sums at fixed points, pairing fixed points at which the linearizations give opposite orientations, we obtain effective circle actions with $2 + 2k$ fixed points on the connected sum $k(S^2 \times S^2)$ for every $k \geq 1$. Now we have $1 + k$ fixed points at which the linearization induces one orientation, and $1 + k$ at which it induces the other orientation. Then making equivariant self-connected sums at pairs of fixed points with linearizations inducing opposite orientations we finally obtain a free circle action on $X_k = k(S^2 \times S^2) \# (1 + k)(S^1 \times S^3)$.

That the Perelman invariant of X_k is infinite follows from the fact that this is so for $S^2 \times S^2$.

If k is odd and large enough, then there are symplectic manifolds Z_k homeomorphic (but not diffeomorphic) to $k(S^2 \times S^2)$, see for example B. Hanke, J. Wehrheim and myself [15]. By the construction given in [15], we may assume that Z_k contains the Gompf nucleus of an elliptic surface. By performing logarithmic transformations inside this nucleus, we can vary the smooth structures on the Z_k in such a way that the number of Seiberg–Witten basic classes with numerical Seiberg–Witten invariant $= \pm 1$ becomes arbitrarily large, cf. Theorem 8.7 of [9] and Example 3.5 of [20].

Consider $Y_k = Z_k \# (1 + k)(S^1 \times S^3)$. This is clearly homeomorphic to X_k . The basic classes on Z_k give rise to monopole classes which are special in the sense of [20]. As their number is unbounded, we have infinitely many distinct smooth structures. By the construction given in [15], the Z_k dissolve after a single stabilization with $S^2 \times S^2$, and therefore the same is true for the Y_k .

As Y_k has non-torsion monopole classes c with $c^2 = 2\chi(Z_k) + 3\sigma(Z_k) = 4 + 4k > 0$, we find that $\overline{\lambda}_{Y_k} \leq -8\pi\sqrt{2(1+k)}$ from Theorem 4. This supremum is not achieved, because Y_k is parallelizable but not flat, and can therefore not carry an Einstein metric by the discussion of the equality case of the Hitchin–Thorpe inequality [16]. Note that Y_k can not collapse with bounded scalar curvature. *A fortiori* it cannot collapse with bounded sectional curvature, and so its minimal volume is strictly positive. \square

Remark 4. That connected sums of manifolds with vanishing minimal volumes may have non-vanishing minimal volumes is immediate by looking at connected sums of tori. The manifolds X_k discussed above have the property that their minimal volumes vanish, although they are connected sums of manifolds with non-vanishing minimal volumes. Thus the minimal volume, and even its (non-)vanishing, does not behave in a straightforward manner under connected sums.

This remark was motivated by the recent preprint of G. Paternain and J. Petean [29]. After an earlier version of Theorem 7 appeared on the arXiv in [21], these authors remarked on the complicated behaviour of the minimal volume under connected sums based on some 6-dimensional examples, see Remark 3.1 in [29].

Remark 5. The insistence on manifolds that dissolve after a single stabilization with $S^2 \times S^2$ was originally motivated by C. T. C. Wall’s theorem [34] showing that every simply connected smooth four-manifold dissolves after some number of stabilizations with $S^2 \times S^2$. On the one hand, because of work of R. Mandelbaum, B. Moishezon and R. Gompf, and also because of [5, 15], we

now know that one stabilization often suffices. On the other hand, there are no examples where it is known that one stabilization does not suffice. Perelman's invariant sheds some unexpected light on this, because after a single stabilization with $S^2 \times S^2$ Perelman's invariant tends to become infinite. In particular, after a single stabilization there are no more non-torsion monopole classes. It is tempting to speculate that the Ricci flow might be useful in investigating the question whether manifolds do indeed always dissolve after a single stabilization.

Going in the opposite direction of Conjecture 1, we have the following:

Theorem 8. *There are homeomorphism types of simply connected four-manifolds that contain infinitely many distinct smooth structures with the same Perelman invariant.*

Proof. The easiest example is furnished by the homeomorphism type underlying the $K3$ surface. This is spin with non-zero signature, and so by the Lichnerowicz argument no manifold in this homeomorphism type admits a metric of positive scalar curvature. However, there are infinitely many smooth structures underlying complex elliptic surfaces in this homeomorphism type. As all these collapse with bounded scalar curvature, their Perelman invariants vanish. \square

In this case the supremum for the Perelman invariant is attained for the standard smooth structure, but not for any of the other ones. This is because the standard smooth structure is the only one admitting an Einstein metric. These were in fact the first examples showing that the existence of an Einstein metric depends on the smooth structure, cf. [19].

Remark 6. If we only look for arbitrarily large numbers of distinct smooth structures, rather than for infinitely many, then we can choose examples with non-vanishing Perelman invariants. For example, V. Braungardt and I proved in [5] that there are arbitrarily large tuples of non-diffeomorphic minimal surfaces of general type with ample canonical bundles. By Theorem 1 above on any such tuple the Perelman invariant is a negative constant. As shown in [5], these examples can be chosen to be spin or non-spin. In the non-spin case they also have infinitely many other, non-complex, smooth structures which, by Theorem 4, have even smaller Perelman invariant.

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